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Quantum groups and the recovery of $U(3)$ symmetry in the Hamiltonian of the nuclear shell model

A Del Sol Mesa†, G Loyola†, M Moshinsky†§, and V Velázquez‡

† Instituto de Física, UNAM, Apartado Postal 20-364, 01000 Mexico, DF Mexico
‡ Instituto de Ciencias Nucleares, UNAM, Apartado Postal 04510, Mexico, DF Mexico

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Abstract. Quantum groups are of current interest because of their applications in many fields of physics. In the present paper we discuss a q-analogue to the Hamiltonian of the nuclear shell model. If q is written as $q = \exp(i\tau)$ where τ is a real number in the interval $0 \leqslant \tau \leqslant 2\pi$, then for $\tau = 0$ we recover the ordinary nuclear shell theory Hamiltonian where the $U(3)$ symmetry is broken by the presence of the spin–orbit coupling term as well as the one depending on L^2. On the other hand if τ is in the interval $0.5 \leqslant \tau \leqslant 2$, the levels corresponding to a given number of quanta N almost collapse to a single one, thus recovering the $U(3)$ symmetry. In the conclusion we compare this result with other procedures to re-establish the $U(3)$ symmetry.

1. Introduction and summary

Quantum algebras have recently been of great interest in physics. The development of the quantum inverse problem method [1] and the study of solutions to the Yang–Baxter equation [2] introduced the notions of quantum groups and algebras.

The growing interest in the quantum groups is related with the similitude of the properties of quantum algebras and those of Lie algebras in connection with both the representation theory [3] and the possible physical applications. The quantum algebra $SU_q(2)$, in particular, has been used for the description of superdeformed bands in even–even nuclei [4], for description of rotational molecular spectra [5], etc.

Recently Biedenharn [6] and Macfarlane [7] independently introduced a q-analogue of the harmonic oscillator and proposed the q-analogue to the Jordan Schwinger map.

Moreover the q-analogue of the standard coherent states, and the q-analogue of the Bargmann representation were studied by various authors [8,9].

In the present paper we shall extend the idea of quantum groups to the Hamiltonian of the nuclear shell model [10]. The appearance of spin–orbit interaction and an L^2 terms in the latter destroys the $U(3)$ symmetry of the original harmonic oscillator. We shall show that this $U(3)$ symmetry can be recovered in a q-deformed version of this Hamiltonian.

We briefly summarize the procedure followed in this paper. In section 2 we discuss the ordinary $U(3) \supset O(3) \supset O(2)$ chain with spin as applied to the Hamiltonian

§ Member of El Colegio Nacional.

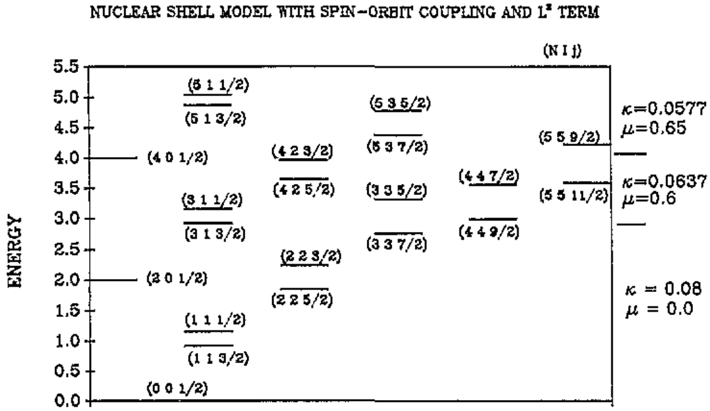

Figure 1. We show the energy levels of the nuclear shell model Hamiltonian given by (2.10). Each level is characterized by the total number of quanta N, and the orbital and total angular momenta given respectively by ℓ and j. Note that the parameters k and μ are functions of N, and shown on the right-hand side of the figure.

of the nuclear shell model and show in figure 1 how the degeneracy associated with $U(3)$ is broken.

In section 3 we introduce the q-deformed operators $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$ as functions of the ordinary creation and annihilation operators η_i, ξ_i, given in the previous section.

In section 4 we discuss the chain $U_q(3) \supset O_q(3) \supset O_q(2)$ of q-deformed groups, with their generators determined explicitly in terms of $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$.

In section 5 we consider the Casimir operators of $U_q(3), O_q(3), O_q(2)$ and show how they give rise to the corresponding commuting integrals of motion $\tilde{N}, \tilde{L}^2, \tilde{L}_3$.

In section 6 we briefly review the q-analogue of total angular momentum j and, in particular, its representation for $j = \frac{1}{2}$, i.e. for the spin, in terms of ordinary Pauli matrices.

In section 7 we discuss the q-deformation of the Hamiltonian of the nuclear shell model and in section 8 its spectra, showing through figures 2 and 3 how the $U(3)$ symmetry is recovered for a certain range of the parameter q.

2. The $U(3) \supset O(3) \supset O(2)$ chain with spin and the Hamiltonian of the nuclear shell model

Let us now consider the ordinary $U(3)$ algebra with the generators E_{ij} expressed in terms of the boson operators η_i, ξ_i, $i = 1,2,3$, i.e.

$$E_{ii} = \eta_i \xi_i \equiv N_i \qquad i = 1,2,3 \tag{2.1a}$$

$$E_{ij} = \eta_i \xi_j \qquad i \neq j \quad i,j = 1,2,3 \tag{2.1b}$$

where η_i, ξ_i and N_i satisfy the following commutation relations:

$$[\xi_i, \eta_j] = \delta_{ij} \tag{2.2a}$$

$$[N_i, \xi_j] = -\delta_{ij}\xi_j \tag{2.2b}$$

$$[N_i, \eta_j] = \delta_{ij}\eta_j \tag{2.2c}$$

$$[N_i, N_j] = [\eta_i, \eta_j] = [\xi_i, \xi_j] = 0\,. \tag{2.2d}$$

Then, we easily obtain

$$[E_{ij}, E_{k\ell}] = E_{i\ell}\delta_{kj} - E_{kj}\delta_{i\ell} \tag{2.3}$$

and the total number operator $N = N_1 + N_2 + N_3 = \eta_1\xi_1 + \eta_2\xi_2 + \eta_3\xi_3$, is a Casimir operator for this algebra, i.e.

$$[N, E_{ij}] = 0 \qquad i, j = 1, 2, 3\,. \tag{2.4}$$

It is well known that the $O(3)$ algebra is a subalgebra of the $U(3)$ one, and the generators of the first one can be defined in the following way:

$$L_1 = -\mathrm{i}(\eta_2\xi_3 - \eta_3\xi_2) \tag{2.5a}$$

$$L_2 = -\mathrm{i}(\eta_3\xi_1 - \eta_1\xi_3) \tag{2.5b}$$

$$L_3 = -\mathrm{i}(\eta_1\xi_2 - \eta_2\xi_1) \tag{2.5c}$$

and they satisfy the commutation relations

$$[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k \tag{2.6}$$

and of course the operator $L^2 = L_1^2 + L_2^2 + L_3^2$ is a Casimir operator. Finally, the $O(2)$ algebra is a subalgebra of the $O(3)$ that has the generator L_3 which is also the Casimir operator of $O(2)$.

Now we are going to talk about the nuclear shell model Hamiltonian, which has been of great interest in nuclear physics [10, 11].

In this model, the nucleons have a common potential of the harmonic oscillator type plus a single-particle spin–orbit coupling term and a term depending on L^2, i.e.

$$H = \boldsymbol{\eta}\cdot\boldsymbol{\xi} - 2k\boldsymbol{L}\cdot\boldsymbol{S} - k\mu L^2 \tag{2.7}$$

where k and μ are parameters taken from the experimental results [10], $\boldsymbol{L}$ and $\boldsymbol{S}$ are respectively the orbital and the spin angular momenta of the nucleon and $L^2 = \boldsymbol{L}\cdot\boldsymbol{L}$.

As is well known the spin $\boldsymbol{S}$ for the nucleons is given by the Pauli matrices

$$S_x = \frac{\sigma_x}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.8a}$$

$$S_y = \frac{\sigma_y}{2} = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{2.8b}$$

$$S_z = \frac{\sigma_z}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.8c}$$

which satisfy the relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k . \tag{2.9}$$

The total angular momentum is $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$, with the eigenvalues of J^2, L^2, S^2 being, respectively, $j(j+1), \ell(\ell+1), \frac{3}{4}$.

Then, the eigenvalues of the nuclear shell model Hamiltonian have the form

$$E(N,\ell,j) = N - k\{j(j+1) - \ell(\ell+1) - \tfrac{3}{4}\} - k\mu\ell(\ell+1) \tag{2.10}$$

where N is the total number of quanta and ℓ and j are the orbital and the total angular momentum respectively.

We can see clearly that the spectrum has lost the symmetry $U(3)$, which corresponds to the harmonic oscillator, as the levels with fixed N and $\ell = N, N-2, \ldots 0$ or 1 and $j = \ell \pm \frac{1}{2}$ are no longer degenerate as seen in figure 1.

3. The q-operators $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$, in terms of the ordinary creation and annihilation operators η_i, ξ_i

The three-dimensional q-harmonic oscillator can be defined in terms of the q-creation operator $\tilde{\eta}_i$, q-annihilation operator $\tilde{\xi}_i = (\tilde{\eta}_i)^+$, and the q-number operator $\tilde{N}_i$, where $(i = 1,2,3)$, and satisfy the following commutation relations [9]:

$$\tilde{\xi}_i\tilde{\eta}_i - q\tilde{\eta}_i\tilde{\xi}_i = q^{-\tilde{N}_i} \tag{3.1a}$$

$$\tilde{\xi}_i\tilde{\eta}_i - q^{-1}\tilde{\eta}_i\tilde{\xi}_i = q^{\tilde{N}_i} \tag{3.1b}$$

$$[\tilde{N}_i, \tilde{\eta}_j] = \delta_{ij}\tilde{\eta}_j \tag{3.1c}$$

$$[\tilde{N}_i, \tilde{\xi}_j] = -\delta_{ij}\tilde{\xi}_j \tag{3.1d}$$

$$[\tilde{\eta}_i, \tilde{\eta}_j] = [\tilde{\xi}_i, \tilde{\xi}_j] = [\tilde{N}_i, \tilde{N}_j] = 0 \tag{3.1e}$$

$$[\tilde{\xi}_i, \tilde{\eta}_j] = 0 \qquad i \neq j . \tag{3.1f}$$

From the relations (3.1*a*, *b*) we can obtain

$$\tilde{\xi}_i\tilde{\eta}_i = [\tilde{N}_i + 1]_q \tag{3.2a}$$

$$\tilde{\eta}_i\tilde{\xi}_i = [\tilde{N}_i]_q \tag{3.2b}$$

where for a given x, we have

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{3.3}$$

where $q = \exp(\tau)$ and τ is a real or purely imaginary number.

The operators $\tilde{N}_i, \tilde{\eta}_i$ and $\tilde{\xi}_i$ act on the q-Hilbert space given by the states [9]

$$|n\rangle_q = \prod_{i=1}^{3} \frac{(\tilde{\eta}_i)^{n_i}|0\rangle_q}{([n_i]_q!)^{1/2}} \tag{3.4}$$

where $|0\rangle_q$ is the q-boson vacuum defined by

$$\tilde{\xi}_i|0\rangle_q = \tilde{N}_i|0\rangle_q = 0 \qquad i = 1,2,3 \tag{3.5a}$$

and the factorial is

$$[n_i]_q! \equiv [n_i]_q[n_i - 1]_q \dots [1]_q \qquad n_i = 0,1,2,3,\dots. \tag{3.5b}$$

The operators act on the basis states in the following manner:

$$\tilde{N}_i|\boldsymbol{n}\rangle_q = n_i|\boldsymbol{n}\rangle_q \tag{3.6a}$$

$$\tilde{\eta}_i|\boldsymbol{n}\rangle_q = [n_i + 1]_q^{1/2}|\boldsymbol{n} + \boldsymbol{e}_i\rangle_q \tag{3.6b}$$

$$\tilde{\xi}_i|\boldsymbol{n}\rangle_q = [n_i]_q^{1/2}|\boldsymbol{n} - \boldsymbol{e}_i\rangle_q \tag{3.6c}$$

where $\boldsymbol{n} \equiv (n_1, n_2, n_3)$ and $\boldsymbol{e}_i$ is a three-dimensional vector with vanishing entries everywhere except for the i component that has value unity.

There is a relation between the q-operators $\tilde{N}_i, \tilde{\eta}_i, \tilde{\xi}_i$ and the usual boson operators η_i, ξ_i [8]:

$$\tilde{N}_i = N_i = \eta_i\xi_i, \tilde{\xi}_i = \left(\frac{[N_i + 1]_q}{N_i + 1}\right)^{1/2} \xi_i, \tilde{\eta}_i = \eta_i\left(\frac{[N_i + 1]_q}{N_i + 1}\right)^{1/2} \tag{3.7}$$

or

$$N_i = \tilde{N}_i, \xi_i = \left(\frac{\tilde{N}_i + 1}{[\tilde{N}_i + 1]_q}\right)^{1/2} \tilde{\xi}_i, \eta_i = \tilde{\eta}_i\left(\frac{\tilde{N}_i + 1}{[\tilde{N}_i + 1]_q}\right)^{1/2}. \tag{3.8}$$

It is easy to show that if we carry out the substitution of the relations (3.8) in the commutation relations for the usual boson operators

$$[\xi_i, \eta_i] = 1 \tag{3.9}$$

we obtain the relations (3.1*a*, *b*).

4. The $U_q(3) \supset O_q(3) \supset O_q(2)$ chain in terms of the q-operators $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$

4.1. The generators of the $U_q(3)$ algebra in terms of the q-operators $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$

The $U_q(3)$ algebra is defined by the generators $\tilde{E}_{ij}$ $(i,j = 1,2,3)$ which can be expressed in terms of the q-operators $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$ [9]:

$$\tilde{E}_{ii} = \tilde{N}_i \qquad i = 1,2,3 \tag{4.1a}$$

$$\tilde{E}_{ii+1} = \tilde{\eta}_i\tilde{\xi}_{i+1} \qquad i = 1,2 \tag{4.1b}$$

$$\tilde{E}_{i+1i} = \tilde{\eta}_{i+1}\tilde{\xi}_i \qquad i = 1,2 \tag{4.1c}$$

$$\tilde{E}_{13} = q^{-\tilde{N}_2}\tilde{\eta}_1\tilde{\xi}_3 \tag{4.1d}$$

$$\tilde{E}_{31} = q^{\tilde{N}_2}\tilde{\eta}_3\tilde{\xi}_1 \tag{4.1e}$$

Using the relations (3.7) we can express the generators of the $U_q(3)$ algebra in terms of those of the $U(3)$, i.e.

$$\tilde{E}_{ii} = E_{ii} = N_i = \eta_i \xi_i \tag{4.2a}$$

$$\tilde{E}_{ii+1} = E_{ii+1}\left(\frac{[N_i+1]_q[N_{i+1}]_q}{(N_i+1)(N_{i+1})}\right)^{1/2} \qquad i=1,2 \tag{4.2b}$$

$$\tilde{E}_{i+1i} = E_{i+1i}\left(\frac{[N_{i+1}+1]_q[N_i]_q}{(N_{i+1}+1)(N_i)}\right)^{1/2} \qquad i=1,2 \tag{4.2c}$$

$$\tilde{E}_{13} = E_{13}\left(\frac{[N_1+1]_q[N_3]_q}{(N_1+1)(N_3)}\right)^{1/2} q^{-N_2} \tag{4.2d}$$

$$\tilde{E}_{31} = E_{31}\left(\frac{[N_3+1]_q[N_1]_q}{(N_3+1)(N_1)}\right)^{1/2} q^{N_2}\,. \tag{4.2e}$$

4.2. The generators of the $O_q(3)$ algebra in terms of $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$

Let consider the quantum algebra $O_q(3)$ generated by the operators $\tilde{L}_+, \tilde{L}_-, \tilde{L}_0$, where $(\tilde{L}_0)^\dagger = \tilde{L}_0$, $(\tilde{L}_+)^\dagger = \tilde{L}_-$ satisfy the commutation relations

$$\begin{aligned}&[\tilde{L}_0, \tilde{L}_\pm] = \pm\tilde{L}_\pm\\ &[\tilde{L}_+\tilde{L}_-] = [2\tilde{L}_0]_q\end{aligned} \tag{4.3}$$

and the $O_q(2)$ algebra has the generator $\tilde{L}_0$.

There is a connection between the generators of the algebra $O_q(3)(\tilde{L}_+, \tilde{L}_0, \tilde{L}_-)$ and the generators of the $O(3)$ algebra (L_+, L_0, L_-) [9, 13]

$$\tilde{L}_+ = L_+\left(\frac{[L+L_0+1]_q[L-L_0]_q}{(L+L_0+1)(L-L_0)}\right)^{1/2} \tag{4.4a}$$

$$\tilde{L}_- = \left(\frac{[L+L_0+1]_q[L-L_0]_q}{(L+L_0+1)(L-L_0)}\right)^{1/2} L_- \tag{4.4b}$$

$$\tilde{L}_0 = L_0 \tag{4.4c}$$

where

$$L \equiv (L^2 + \tfrac{1}{4})^{1/2} - \tfrac{1}{2}\,. \tag{4.4d}$$

Carrying out the substitution of the relations (4.4*a, b, c*) in the commutation relations for the $O(3)$ generators

$$[L_0, L_\pm] = \pm L_\pm \tag{4.5a}$$

$$[L_+, L_-] = 2L_0 \tag{4.5b}$$

we obtain the relations (4.3).

As is well known, the generators L_+, L_-, L_0 can be expressed in terms of the boson operators η_i, ξ_i $(i = 1,2,3)$

$$L_+ \boxminus (L_1 + iL_2) = -i\{\eta_2\xi_3 - \eta_3\xi_2 + i\eta_3\xi_1 - i\eta_1\xi_3\} \tag{4.6a}$$

$$L_- \boxminus (L_1 - iL_2) = -i\{\eta_2\xi_3 - \eta_3\xi_2 - i\eta_3\xi_1 + i\eta_1\xi_3\} \tag{4.6b}$$

$$L_0 \boxminus L_3 = -i\{\eta_1\xi_2 - \eta_2\xi_1\}\,. \tag{4.6c}$$

On the other hand we know the relation between the operators η_i, ξ_i, N_i and $\tilde{\eta}_i\tilde{\xi}_i, \tilde{N}_i$ (equations 3.8) and the relation of the generators L_+, L_0, L_- with those $\tilde{L}_+, \tilde{L}_0, \tilde{L}_-$ (equations (4.4*a*–*c*)).

Thus we can express the generators of the $O_q(3)$ algebra $(\tilde{L}_+, \tilde{L}_0, \tilde{L}_-)$ in terms of the q-boson operators $\tilde{\xi}_i, \tilde{\eta}_i, \tilde{N}_i$ $(i = 1,2,3)$:

$$\begin{aligned}\tilde{L}_+ = (-i)\{&\tilde{\eta}_2\tilde{\xi}_3 G(\tilde{N}_2, \tilde{N}_3) - \tilde{\eta}_3\tilde{\xi}_2 G(\tilde{N}_3, \tilde{N}_2) + i\tilde{\eta}_3\tilde{\xi}_1 G(\tilde{N}_3, \tilde{N}_1)\\ &- i\tilde{\eta}_1\tilde{\xi}_3 G(\tilde{N}_1, \tilde{N}_3)\} f(L_0, L)\end{aligned} \tag{4.7a}$$

$$\tilde{L}_- = (\tilde{L}_+)^\dagger \tag{4.7b}$$

$$\tilde{L}_0 = L_0 = -i\{\tilde{\eta}_1\tilde{\xi}_2 G(\tilde{N}_1, \tilde{N}_2) - \tilde{\eta}_2\tilde{\xi}_1 G(\tilde{N}_2, \tilde{N}_1)\} \tag{4.7c}$$

where

$$G(\tilde{N}_i, \tilde{N}_j) \equiv \left(\frac{(\tilde{N}_i + 1)(\tilde{N}_j)}{[\tilde{N}_i + 1]_q [\tilde{N}_j]_q}\right)^{1/2} \tag{4.7d}$$

and

$$f(L_0, L) \equiv \left(\frac{[L + L_0 + 1]_q [L - L_0]_q}{(L + L_0 + 1)(L - L_0)}\right)^{1/2}. \tag{4.7e}$$

Note from (4.4*d*) and (2.5) that L and L_0 can be expressed in terms of η_i, ξ_i. As these in turn are functions of $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$, as seen in (3.8), we can finally obtain $f(L, L_0)$ of (4.7) in terms of the latter. Thus $\tilde{L}_\pm, \tilde{L}_0$ are explicit functions of $\tilde{\eta}_i, \tilde{\xi}_i, \tilde{N}_i$.

5. The Casimir operator of $U_q(3)$, $O_q(3)$ and $O_q(2)$ algebras and the integrals of motion

In connection with the chain $U_q(3) \supset O_q(3) \supset O_q(2)$ we can find three commuting integrals of motion

$$\tilde{N} = \tilde{N}_1 + \tilde{N}_2 + \tilde{N}_3 \tag{5.1a}$$

$$\tilde{L}^2 \equiv C_2^q = \tilde{L}_-\tilde{L}_+ + [\tilde{L}_0 + \tfrac{1}{2}]_q^2 - \tfrac{1}{4} \tag{5.1b}$$

$$\tilde{L}_0 = \tilde{L}_3\,. \tag{5.1c}$$

The total number of quanta operator $\tilde{N}$ is a Casimir of the $U_q(3)$ algebra, i.e.

$$[\tilde{N}, \tilde{E}_{ij}] = 0 \qquad i, j = 1, 2, 3 \tag{5.2}$$

and the operators $\tilde{L}^2$ and $\tilde{L}_0$ are Casimirs of the $O_q(3)$ and $O_q(2)$ algebras respectively, i.e.

$$[\tilde{L}^2, \tilde{L}_0] = 0 \tag{5.3a}$$

$$[\tilde{L}^2, \tilde{L}_\pm] = 0\,. \tag{5.3b}$$

All these operators commute each other. We can clearly see that $[\tilde{N}, \tilde{L}_0] = 0$, which is easy to show because we know that $\tilde{N} = N$ from (3.7*a*) and $\tilde{L}_0 = L_0$ from (4.4*c*), so then we have

$$[\tilde{N}, \tilde{L}_0] = [N, L_0] = 0\,. \tag{5.4}$$

It is also clear that $[\tilde{L}_0, \tilde{L}^2] = 0$ because $\tilde{L}^2$ is a Casimir operator of the $O_q(3)$ algebra.

To prove the relation $[\tilde{N}, \tilde{L}^2] = 0$, it is useful to note that

$$[\tilde{N}, \tilde{L}^2] = [\tilde{N}, \tilde{L}_- \tilde{L}_+] = \left[N, L_- L_+ \left(\frac{[L + L_0 + 1]_q [L - L_0]_q}{(L + L_0 + 1)(L - L_0)}\right)\right] = 0 \tag{5.5}$$

where in (5.5) we used the relations (3.7) and (4.4*a*, *b*).

6. The discussion of the spin part and the states for the total angular momentum

Let us consider now the quantum algebra $SU_q(2)$ generated by the operators $\tilde{J}_+, \tilde{J}_0, \tilde{J}_-$ satisfying the relations

$$[\tilde{J}_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm \tag{6.1a}$$

$$[\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_0]_q\,. \tag{6.1b}$$

The irreducible representations are given by the vectors of the Hilbert space $|jm\rangle_q$, where j may take the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m = -j, -j+1, \ldots, j$ such that [14]

$$\tilde{J}_0 |jm\rangle_q = m |jm\rangle_q \tag{6.2a}$$

$$\tilde{J}_\pm |jm\rangle_q = \left([j \mp m]_q [j \pm m + 1]_q\right)^{1/2} |jm \pm 1\rangle_q\,. \tag{6.2b}$$

In the case of the spin operators $\tilde{S}_0, \tilde{S}_+, \tilde{S}_-$, where $s = \frac{1}{2}$, we have

$$\tilde{S}_0 |\tfrac{1}{2}, m\rangle_q = m |\tfrac{1}{2}, m\rangle_q \tag{6.3a}$$

$$\tilde{S}_\pm |\tfrac{1}{2}, m\rangle_q = \left(\left[\tfrac{1}{2} \mp m\right]_q \left[\tfrac{1}{2} \pm m + 1\right]_q\right)^{1/2} |\tfrac{1}{2}, m \pm 1\rangle_q\,. \tag{6.3b}$$

The matrix elements are in the form

$$\left\langle \tfrac{1}{2},-\tfrac{1}{2}\middle|\tilde{S}_0\middle|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_q = -\tfrac{1}{2} \tag{6.4a}$$

$$\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|\tilde{S}_0\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_q = \tfrac{1}{2} \tag{6.4b}$$

$$\left\langle \tfrac{1}{2},-\tfrac{1}{2}\middle|\tilde{S}_-\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_q = 1 \tag{6.4c}$$

$$\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|\tilde{S}_+\middle|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_q = 1 \tag{6.4d}$$

and the rest of them are equal to zero.

Then, in matrix form

$$\tilde{S}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \tilde{S}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tilde{S}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.5}$$

It is clear, that in the case $s = \frac{1}{2}$, the matrix elements of the generators $(\tilde{S}_+, \tilde{S}_0, \tilde{S}_-)$ in the basis $\left|\frac{1}{2}, m\right\rangle_q$ are identical to the matrix elements of the generators of the $SU(2)$ algebra (S_+, S_0, S_-) in the usual basis $\left|\frac{1}{2}, m\right\rangle$. This is a special case for the $SU_q(2)$ algebra.

Finally, we would like to note that from the basis vectors $|j_1 m_1\rangle_q |j_2 m_2\rangle_q$ of the direct product $\mathcal{D}^{j_1} \otimes \mathcal{D}^{j_2}$ of two irreducible representations we can obtain linear combinations

$$|j_1 j_2; jm\rangle_q = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | jm\rangle_q |j_1 m_1\rangle_q |j_2 m_2\rangle_q \tag{6.6}$$

which are basis vectors of the irreducible representations of this algebra.

The coefficients $\langle j_1 m_1 j_2 m_2 | jm\rangle_q$ are the q-analogue of the Clebsch–Gordan coefficients [14], and we are interested in the particular case when $j_1 = \ell,\ j_2 = \frac{1}{2}$.

7. The q-deformation of the Hamiltonian of the nuclear shell model

The q-deformed Hamiltonian of the nuclear shell model has the form

$$\tilde{H}_0 = \tilde{N} - k\{\tilde{J}^2 - \tilde{L}^2 - \tilde{S}^2\} - k\mu\tilde{L}^2 \tag{7.1}$$

where $\tilde{N}$ is the q-number operator (3.1), $\tilde{L}, \tilde{S}$ and $\tilde{J}$ are respectively the q-analogues of the orbital, spin and total angular momentum operators, (4.3), (6.2), (6.3).

In this case, we have the eigenvalues in the following manner:

$$\tilde{E} = N - k\left\{\left[j+\tfrac{1}{2}\right]_q^2 - \left[\ell+\tfrac{1}{2}\right]_q^2 - \tfrac{3}{4}\right\} - k\mu\left\{\left[\ell+\tfrac{1}{2}\right]_q^2 - \tfrac{1}{4}\right\} \tag{7.2}$$

where N is again the total number of quanta, ℓ and j are the orbital and the total angular momenta respectively and k and μ are parameters taken from experimental results. We note that for the case $s = \frac{1}{2}$ the eigenvalue of the operator $\tilde{S}^2$ is equal to $\frac{3}{4}$ (section 6).

8. Spectra of the q-deformed Hamiltonian and recovery of the $U(3)$ symmetry

As we showed in section 7 the eigenvalues of the q-deformed nuclear shell model Hamiltonian are given by (7.2). Then if we choose the value of q in the form $q = e^{i\tau}$, we have

$$\begin{aligned}\bar{E}(N,\ell,j) &= N - k\{[j+\tfrac{1}{2}]_q^2 - [\ell+\tfrac{1}{2}]_q^2 - \tfrac{3}{4}\} - k\mu\{[\ell+\tfrac{1}{2}]_q^2 - \tfrac{1}{4}\} \\ &= N - k\left\{\frac{\sin^2(j+\frac{1}{2})\tau}{\sin^2\tau} - \frac{\sin^2(\ell+\frac{1}{2})\tau}{\sin^2\tau} - \frac{3}{4}\right\} \\ &\quad - k\mu\left\{\frac{\sin^2(\ell+\frac{1}{2})\tau}{\sin^2\tau} - \frac{1}{4}\right\}.\end{aligned} \tag{8.1}$$

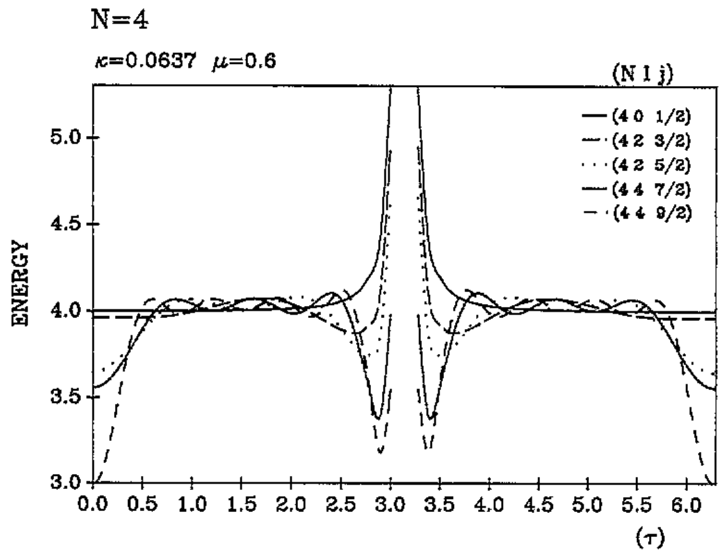

Figure 2. We show the energy levels $E(N,\ell,j)$ of the q-deformed nuclear shell model Hamiltonian of (8.1) for $N = 4$, as function of the parameter τ in the interval $0 \leqslant \tau \leqslant 2\pi$. For $\tau = 0$ the energy levels are the same as those of figure 1 when $N = 4$. In the intervals $0.5 \leqslant \tau \leqslant 2.0$ and $4.0 \leqslant \tau \leqslant 5.5$, the levels, characterized by N, ℓ, j, become almost degenerate. At $\tau = \pi$ they go to infinity because $\sin^2\tau$ in the denominator vanishes. The periodicity of $E(N,\ell,j)$ with respect to τ is 2π.

One interesting result is the fact that the symmetry $U(3)$ is almost recovered when we introduce the q-deformation in the nuclear shell model Hamiltonian, when we are in the interval of $0.5 \leqslant \tau \leqslant 2.0$ and $4.0 \leqslant \tau \leqslant 5.5$ as shown in figures 2 and 3 where we considered the important shells for medium and heavy nuclei, i.e. $N = 4, 5$.

What the figures tell us is that the levels broken by the normal spin–orbit coupling and the term L^2, which are given in figures 2 and 3 when $\tau = 0$, become almost degenerate in the intervals $0.5 \leqslant \tau \leqslant 2$, $4.0 \leqslant \tau \leqslant 5.5$. For $\tau = \pi$ the levels diverge

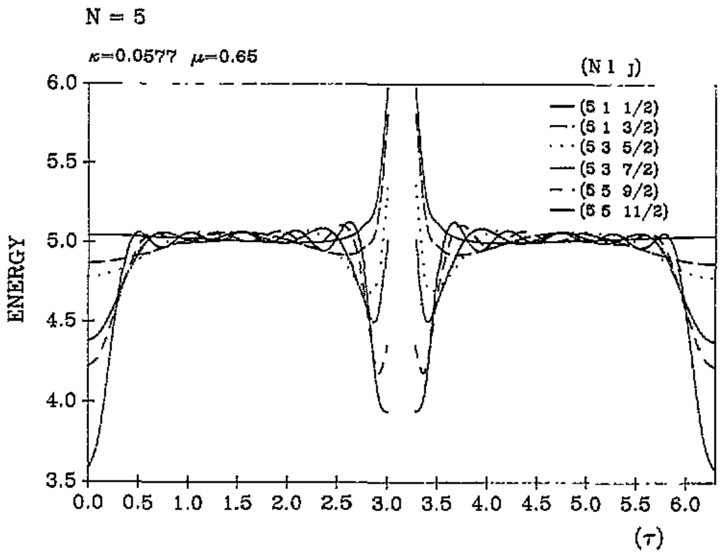

Figure 3. We show the energy levels $E(N, \ell, j)$ of the q-deformed nuclear shell model Hamiltonian of (8.1) for $N = 5$, as function of the parameter τ in the interval $0 \leqslant \tau \leqslant 2\pi$. For $\tau = 0$ the energy levels are the same as those of figure 1 when $N = 5$. In the intervals $0.5 \leqslant \tau \leqslant 2.0$ and $4.0 \leqslant \tau \leqslant 5.5$, the levels, characterized by N, ℓ, j, become almost degenerate. At $\tau = \pi$ they go to infinity because $\sin^2 \tau$ in the denominator vanishes. The periodicity of $E(N, \ell, j)$ with respect to τ is 2π.

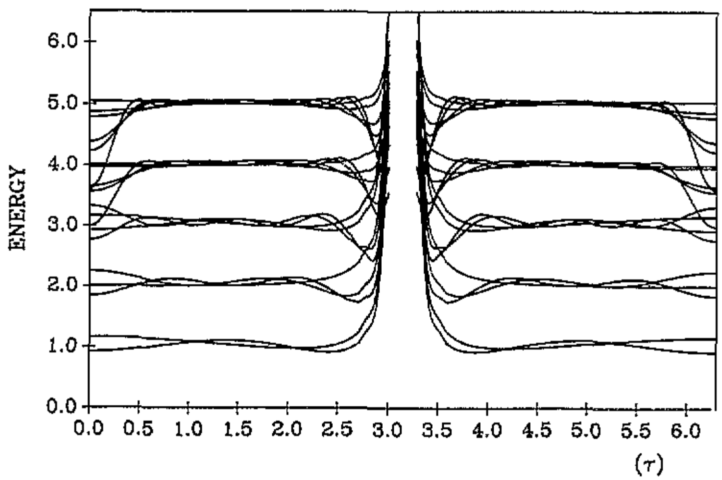

Figure 4. We show the energy levels, of the q-deformed nuclear shell model Hamiltonian of (8.1) for $N = 0, 1, 2, 3, 4, 5$, as functions of the parameter τ in the interval $0 \leqslant \tau \leqslant 2\pi$. It has the same characteristics as in figures 2 and 3.

to infinity because $\sin^2 \tau$ in the denominator of (8.1) vanishes. At $\tau = 2\pi$ the levels return to their original values at $\tau = 0$, and from then on the graph is reproduced periodically. We also include in figure 4 all the levels up to $N = 5$ as function of τ in the interval $0 \leqslant \tau \leqslant 2\pi$.

The behaviour reported in the previous paragraphs has already been observed by Gupta *et al* [15] in the much simpler chain $U_q(2) \supset O_q(2)$. In our notation they considered first the ordinary two-dimensional Hamiltonian

$$
\begin{aligned}
H &= (\eta_1\xi_1 + \eta_2\xi_2) + kM^2 \\
M &= \eta_1\xi_2 + \eta_2\xi_1
\end{aligned}
\tag{8.2}
$$

whose eigenvalues are

$$E(N, m) = N + km^2 \tag{8.3}$$

where N is the number of quanta and $m = N, N-2, \ldots, 1$ or 0. They then proceeded to deform it so the eigenvalues become

$$\bar{E}(N, m) = N + k[m]_q^2 \tag{8.4}$$

following the same type of analysis that takes us from (2.10) to (8.1). In their figure 1(*b*), they take $q = \exp(\mathrm{i}\tau)$ with $0 \leqslant \tau \leqslant 1$ and it shows exactly the same type of behaviour as our figures 2 and 3, i.e. the levels well separated at $\tau = 0$ become almost degenerate for τ close to 1. This corroborates our analysis within a much simpler, and less physical, example.

We note also that Gupta *et al* [15] have discussed the case when q is real where the separation is enhanced instead of diminished when q increases. This also happens in our paper and that is the reason why we do not discuss here the behaviour of $\bar{E} = (N, l, j)$ of (8.1) for real q.

9. Conclusion

The main result of our paper is that in a q-deformed Hamiltonian for the nuclear shell model we can almost recover the $U(3)$ symmetry for some values of the parameter τ related to the q by $q = \exp(\mathrm{i}\tau)$.

The recovery of the $U(3)$ symmetry can also be achieved by other means. For example, Castaños *et al* [12] proposed applying the operator

$$U = 2(\boldsymbol{\xi} \cdot \boldsymbol{S})(\boldsymbol{\eta} \cdot \boldsymbol{\xi} - 2\boldsymbol{L} \cdot \boldsymbol{S})^{-1/2} \tag{9.1}$$

to the Hamiltonian of the nuclear shell model given in (2.7), where the symbols are defined in section 2.

The new Hamiltonian of the nuclear shell model is then given by [12]

$$\bar{H} = UHU^{+} = \boldsymbol{\eta} \cdot \boldsymbol{\xi} + 1 - 2k(2\mu - 1)\boldsymbol{L} \cdot \boldsymbol{S} - k\mu L^2 - 2k(\mu - 1) \tag{9.2}$$

and so if $\mu = \frac{1}{2}$ the spin–orbit coupling disappears and the energy levels could be characterized by the eigenvalues of the Casimir operators in the chain $U(3) \supset O(3)$.

The q-analogue Hamiltonian is then an alternative to the procedure indicated in the previous paragraphs, and could possibly be of practical interest in calculations of structures of medium and heavy nuclei, in a way similar to what was done by Elliott [16] in his analysis of nuclei in the 2s1d shell, in which he made use of the almost $U(3)$ symmetry in that shell.

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